

# Fermions in Self-dual Vortex Background on a String-like Defect

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## Abstract

By using the self-dual vortex background on extra two-dimensional Riemann surfaces in 5+1 dimensions, the localization mechanism of bulk fermions on a string-like defect with the exponentially decreasing warp-factor is obtained. We give the conditions under which localized spin 1/2 and 3/2 fermions can be obtained.

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## I. INTRODUCTION

It is now widely believed that extra dimensions play an important role in constructing a unified theory of all interactions and provides us with a new solution to hierarchy problem [1, 2]. The possible existence of such dimensions got strong motivation from theories that try to incorporate gravity and gauge interactions in a unique scheme, in a reliable manner. The idea dates back to the 1920's, to the works of Kaluza and Klein [3] who tried to unify electromagnetism with Einstein gravity by assuming that the photon originates from the fifth component of the metric.

Recently, co-dimension two models in six dimensions have been a topic of increasing interest [4, 5, 6, 7]. Apart from model construction, the question of solving the cosmological constant problem has been the primary issue addressed in several articles [8]. Other aspects such as cosmology, brane gravity [9], fermion families and chirality [10] etc. have been discussed by numerous authors. A list of some recent articles on codimension two models is provided in Ref. [11].

In the brane world scenario, our universe is regarded as a 3-brane embedded in a higher-dimensional space-time with non-factorizable warped geometry. It is a priori assumed that all the matter fields are constrained to live on the three brane, whereas gravity is free to propagate in the extra dimension. Then a key ingredient for realizing the brane world idea is localization of various bulk fields on a brane by a natural mechanism. In other words, in this scenario various fields we observe in our universe are nothing but the zero modes of the corresponding bulk fields which are trapped on our brane by some ingenious mechanism.

This localization mechanism has been recently investigated within the framework of a local field theory. Ever since Goldberger and Wise [12] added a bulk scalar field to fix the location of the branes in five dimensions, investigations with bulk fields became an active area of research. It has been shown that the graviton [2] and the massless scalar field [13] have normalizable zero modes on branes of different types, that the Abelian vector fields are not localized in the Randall-Sundrum (RS) model in five dimensions but can be localized in some higher-dimensional generalizations of it [6]. In contrast, in [13, 14] it was shown that fermions do not have normalizable zero modes in five dimensions, while in [6] the same result was derived for a compactification on a string [7] in six dimensions. Subsequently, Randjbar-Daemi *et al* studied localization of bulk fermions on a brane with inclusion of scalar

backgrounds [15] and minimal gauged supergravity [16] in higher dimensions and gave the conditions under which localized chiral fermions can be obtained.

Since spin half fields can not be localized on the brane [2, 6] in five or six dimensions by gravitational interaction only, it becomes necessary to introduce additional non-gravitational interactions to get spinor fields confined to the brane or string-like defect. Fermionic zero modes in the absence of gravity and in four dimensions in vortex background were studied in [17] and extended to the case of six dimensional space-times in gravity, gauge and vortex backgrounds in [18]. The aim of the present article is to study localization of bulk fermions on a string-like defect with codimension 2 in self-dual vortex background. In this article, we first review the solutions to Einstein's equations with a warp factor in a 6-dimensional space-time, which has been studied by many groups [4, 7, 19, 20]. Then, we shall prove that spin 1/2 and 3/2 fields can be localized on a defect with the exponentially decreasing warp factor if the self-dual vortex and gravitational backgrounds are considered.

## II. SELF-DUAL VORTEX ON A TWO-DIMENSIONAL CURVED SPACE

This paper is focused on braneworld models with codimension greater than one. In particular, we shall be exclusively concerned with bulk spacetimes in six dimensions generically represented by the line element

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N \\ &= g_{\mu\nu}(x, y) dx^\mu dx^\nu + \gamma_{ij}(y) dy^i dy^j, \end{aligned} \quad (1)$$

where  $M, N$  denote 6-dimensional space-time indices,  $\mu, \nu = 0, 1, 2, 3$  and  $i, j = 1, 2$  for our 4-dimensional space-time and the two dimensional extra space  $K^2$ , respectively,  $\gamma_{ij}$  is the metric on  $K^2$ .

To generate the vortex solution, we introduce the generalized Abelian Higgs Lagrangian

$$\mathcal{L}_{AH} = \sqrt{-g} \left( -\frac{1}{4} F_{MN} F^{MN} + (D^M \phi)^\dagger (D_M \phi) - \frac{\lambda}{2} (\|\phi\|^2 - v^2)^2 \right), \quad (2)$$

where  $g = \det(g_{MN})$ ,  $F_{MN} = \partial_M A_N - \partial_N A_M$ ,  $\phi = \phi(y^k)$  is a complex scalar field on extra dimensions,  $\|\phi\| = (\phi\phi^*)^{\frac{1}{2}}$ ,  $A_M$  is a U(1) gauge field,  $D_M = \partial_M - ieA_M$  is gauge-covariant. In Eq. (2),  $v$  is the vacuum expectation value of the Higgs field determining the masses of the Higgs and of the gauge boson

$$m_H = \sqrt{2\lambda}v, \quad m_V = ev. \quad (3)$$

The Abrikosov-Nielsen-Olesen vortex solution on  $K^2$  could be generated from the Higgs field  $\phi$ . In the generalized Abelian Higgs model, if the system admits a Bogomol'nyi limit [21], one can arrive at the first-order Bogomol'nyi self-duality equations in a curved space [22]:

$$B = \mp e(\|\phi\|^2 - v^2), \quad (4)$$

$$D_i \phi \mp i\sqrt{\gamma}\epsilon_{ij}\gamma^{jk}D_k \phi = 0. \quad (5)$$

The complex Higgs field  $\phi$  can be regarded as the complex representation of a two-dimensional vector field  $\vec{\phi} = (\phi^1, \phi^2)$  over the base space, it is actually a section of a complex line bundle on the base manifold. Substituting  $\phi = \phi^1 + i\phi^2$  and  $D_i = \partial_i - ieA_i$  into Eq. (5) and splitting the real part from the imaginary part, we obtain two equations

$$\partial_i \phi^j = -eA_i \epsilon_{jk} \phi^k \mp \sqrt{\gamma} \epsilon_{il} \gamma^{lm} (\epsilon_{jk} \partial_m \phi^k - eA_m \phi^j), \quad (6)$$

From Eq. (6), by calculating  $\partial_i \phi^* \phi - \partial_i \phi \phi^*$ , we can obtain the expression of the gauge potential

$$eA_i = -\frac{1}{2i\|\phi\|^2} (\partial_i \phi^* \phi - \partial_i \phi \phi^*) \mp \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} \partial_k \ln \|\phi\|. \quad (7)$$

If we define the unit vector

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad (a, b = 1, 2) \quad (8)$$

and note the identity

$$\epsilon_{ab} n^a \partial_i n^b = \frac{1}{2i\|\phi\|^2} (\partial_i \phi^* \phi - \partial_i \phi \phi^*), \quad (9)$$

Eq. (7) further simplifies to:

$$eA_i = -\epsilon_{ab} n^a \partial_i n^b \mp \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} \partial_k \ln \|\phi\|. \quad (10)$$

In curved space, the magnetic field is defined by  $B = -\frac{1}{\sqrt{\gamma}} \epsilon^{ij} \partial_i A_j$ , according to Eq. (10), we have

$$e\sqrt{\gamma}B = \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b \pm \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|). \quad (11)$$

So the first self-duality equation (4) can be generalized to

$$\mp e^2 \sqrt{\gamma} (\|\phi\|^2 - v^2) = \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b \pm \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|). \quad (12)$$

According to Duan's  $\phi$ -mapping topological current theory [23], it is easy to see that the first term on the RHS of Eq. (12) bears a topological origin, and the topological term just

describes the non-trivial distribution of  $\vec{n}$  [24]. Noticing  $\partial_i n^a = \partial_i \phi^a / \|\phi\| + \phi^a \partial_i (1/\|\phi\|)$  and the Green function relation in  $\phi$ -space:  $\partial_a \partial_a \ln(\|\phi\|) = 2\pi \delta^2(\vec{\phi})$ , ( $\partial_a = \frac{\partial}{\partial \phi^a}$ ), it can be proved that [25]

$$\epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b = 2\pi \delta^2(\vec{\phi}) J\left(\frac{\phi}{y}\right) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k), \quad (13)$$

where  $J(\phi/y)$  is the Jacobian and  $W_k = \beta_k \eta_k$  is the winding number around the  $k$ -th vortex, the positive integer  $\beta_k$  is the Hopf index and  $\eta_k = \pm 1$  is the Brouwer degree,  $\vec{y}_k$  are the coordinates of the  $k$ -th vortex. So the first Bogomol'nyi self-duality equation (4) should be

$$\mp e^2 \sqrt{\gamma} (\|\phi\|^2 - v^2) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \mp \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|). \quad (14)$$

Obviously the first term on the RHS of Eq. (14) describes the topological self-dual vortex.

Now let us discuss the case of flat space for the self-duality equation (14). In this special case,  $\gamma_{ij} = \delta_{ij}$  and Eq. (14) reads as

$$\mp e^2 (\|\phi\|^2 - v^2) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \mp \partial_i \partial_i \ln \|\phi\|. \quad (15)$$

While the corresponding conventional self-duality equation is [26]

$$e^2 (\|\phi\|^2 - v^2) = \partial_i \partial_i \ln \|\phi\|. \quad (16)$$

Comparing our equation (15) with Eq. (16), one can see that the topological term  $2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k)$ , which describes the topological self-dual vortex, is missed in the conventional equation. Obviously, only when the field  $\phi \neq 0$ , the topological term vanishes and the conventional equation is correct. So, the exact self-duality equation should be Eq. (15) for flat space and Eq. (14) for curved one. As for conventional self-dual nonlinear equation (16), a great deal of work has been done by many physicists on it, and a vortex-like solution was given by Jaffe [27]. But no exact solutions are known.

In the following sections, we first give a brief review of a string-like defect solution to Einstein's equations with sources, then we study fermionic zero modes coupled with the vortex background and the localization of fermions on the string-like defect with an exponentially decreasing warp factor.

### III. REVIEW OF A STRING-LIKE DEFECT

Let us consider Einstein's equations with a bulk cosmological constant  $\Lambda$  and an energy-momentum tensor  $T_{MN}$  in general six dimensions:

$$R_{MN} - \frac{1}{2}g_{MN}R = -\Lambda g_{MN} + \kappa_6^2 T_{MN}, \quad (17)$$

where  $\kappa_6$  denotes the 6-dimensional gravitational constant with a relation  $\kappa_6^2 = 8\pi G_N = 8\pi/M_*^4$ ,  $G_N$  and  $M_*$  being the 6-dimensional Newton constant and the 6-dimensional Planck mass scale, respectively, the energy-momentum tensor is defined as

$$T_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{MN}} \int d^6x \sqrt{-g} L_m. \quad (18)$$

We shall consider the most general metric ansatz for a warped brane embedded in six dimensions obeying four dimensional Poincare invariance

$$ds^2 = e^{-A(r)} \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu + dr^2 + R_0^2 e^{-B(r)} d\theta^2, \quad (19)$$

where the radial coordinate  $r$  is infinitely extended ( $0 < r < \infty$ ) and the compact coordinate  $\theta$  ranges from  $0 \leq \theta \leq 2\pi$ ,  $R_0$  is an additional parameter characterizing the extra compact direction. Moreover, we shall adopt the ansatz for the energy-momentum tensor respecting the spherical symmetry:

$$T_\nu^\mu = \delta_\nu^\mu t_0(r), \quad T_r^r = t_r(r), \quad T_\theta^\theta = t_\theta(r), \quad (20)$$

where  $t_i (i = 0, r, \theta)$  are functions of only the radial coordinate  $r$ .

Under these ansatzs, Einstein's equations (17) and the conservation law for energy-momentum tensor  $\nabla^M T_{MN} = 0$  reduce to

$$e^A \hat{R} - 3(A')^2 - 2A'B' - 2\Lambda + 2\kappa_6^2 t_r = 0, \quad (21)$$

$$e^A \hat{R} + 4A'' - 5(A')^2 - \frac{1}{2}(B')^2 - 2\Lambda + 2\kappa_6^2 t_\theta = 0, \quad (22)$$

$$e^A \hat{R} + 2B'' + 6A'' - 10(A')^2 - (B')^2 - 4\Lambda + 4\kappa_6^2 t_0 = 0, \quad (23)$$

$$t'_r = 2A'(t_r - t_0) + \frac{1}{2}B'(t_r - t_\theta), \quad (24)$$

where  $\hat{R}$  are the scalar curvatures associated with the metric  $\hat{g}_{\mu\nu}$ , and the prime denotes the derivative with respect to  $r$ . Here we define the cosmological constant  $\hat{\Lambda}$  on the 3-brane by the equation

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} = -\hat{\Lambda}\hat{g}_{\mu\nu}. \quad (25)$$

It is now known that there are many interesting solutions to these equations (see, for instance, [19]). Here, we shall consider the brane solutions with a warp factor

$$A(r) = cr, \quad (26)$$

where  $c$  is a constant. A specific solution occurs when we have the spontaneous symmetry breakdown  $t_r = -t_\theta$  [19]:

$$ds^2 = e^{-cr} \hat{g}_{\mu\nu} dx^\mu dx^\nu + dr^2 + R_0^2 e^{-c_1 r} d\theta^2, \quad (27)$$

where

$$c^2 = \frac{2}{5}(\kappa_6^2 t_\theta - \Lambda) > 0, \quad (28)$$

$$c_1 = c - \frac{2}{c} \kappa_6^2 t_\theta, \quad (29)$$

$$\hat{R} = 4\hat{\Lambda} = 0. \quad (30)$$

This special solution would be utilized to analyze localization of fermionic fields on a string-like defect.

#### IV. LOCALIZATION OF FERMIONS

In this section, we have the physical setup in mind such that ‘local cosmic string’ sits at the origin  $r = 0$  and then ask the question of whether various bulk fermions with spin 1/2 and 3/2 can be localized on the brane with the exponentially decreasing warp factor by means of the gravitational interaction and vortex backgrounds. Of course, we have implicitly assumed that various bulk fields considered below make little contribution to the bulk energy so that the solution (27) remains valid even in the presence of bulk fields.

##### A. Spin 1/2 fermionic field

In this subsection we study localization of a spin 1/2 fermionic field in gravity (27) and vortex backgrounds. It will be shown that provided that if the vortex background satisfies certain condition, there is a localized zero mode on the string-like defect.

Let us consider the Dirac action of a massless spin 1/2 fermion coupled to gravity and vortex backgrounds:

$$S_m = \int d^D x \sqrt{-g} i \bar{\Psi} \Gamma^M D_M \Psi, \quad (31)$$

from which the equation of motion is given by

$$\Gamma^M(\partial_M + \omega_M - ieA_M)\Psi = 0, \quad (32)$$

where  $\omega_M = \frac{1}{4}\omega_M^{\bar{M}\bar{N}}\Gamma_{\bar{M}}\Gamma_{\bar{N}}$  is the spin connection with  $\bar{M}, \bar{N}, \dots$  denoting the local Lorentz indices,  $\Gamma^M$  and  $\Gamma^{\bar{M}}$  are the curved gamma matrices and the flat ones, respectively. From the formula  $\Gamma^M = e_M^{\bar{M}}\Gamma^{\bar{M}}$  with  $e_M^{\bar{M}}$  being the vielbein, we have the relations:

$$\Gamma^\mu = e^{\frac{1}{2}cr}\hat{e}_\mu^\mu\Gamma^{\bar{\mu}}, \quad \Gamma^r = \delta_r^{\bar{r}}\Gamma^{\bar{r}}, \quad \Gamma^\theta = R_0^{-1}e^{\frac{1}{2}c_1r}\delta_\theta^{\bar{\theta}}\Gamma^{\bar{\theta}}. \quad (33)$$

The spin connection  $\omega_M^{\bar{M}\bar{N}}$  in the covariant derivative  $D_M\Psi$  is defined as

$$\begin{aligned} \omega_M^{\bar{M}\bar{N}} &= \frac{1}{2}e^{N\bar{M}}(\partial_M e_N^{\bar{N}} - \partial_N e_M^{\bar{N}}) \\ &\quad - \frac{1}{2}e^{N\bar{N}}(\partial_M e_N^{\bar{M}} - \partial_N e_M^{\bar{M}}) \\ &\quad - \frac{1}{2}e^{P\bar{M}}e^{Q\bar{N}}(\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}})e_M^{\bar{R}}. \end{aligned} \quad (34)$$

So the non-vanishing components of  $\omega_M$  are

$$\omega_\mu = \frac{1}{4}c\Gamma_r\Gamma_\mu, \quad \omega_\theta = \frac{1}{4}c_1\Gamma_r\Gamma_\theta. \quad (35)$$

In what follows, to illustrate how the vortex background affects the fermionic zero modes, we first discuss the simple case that the Higgs field  $\phi$  is only relative to  $r$ , and then solve the general Dirac equation for the vacuum Higgs field solution  $\|\phi\| = v$ .

Case I:  $\phi = \phi(r) = \phi^1(r) + i\phi^2(r)$ .

In this case, Eq. (10) reduces to:

$$eA_r = -\epsilon_{ab}n^a\partial_r n^b, \quad (36)$$

$$eA_\theta = \pm R_0^{-1}e^{\frac{1}{2}c_1r}\partial_r \ln \|\phi\|. \quad (37)$$

The Dirac equation then becomes

$$\left\{ e^{\frac{1}{2}cr}\hat{e}_\mu^\mu\Gamma^{\bar{\mu}}\hat{D}_\mu + \Gamma^r\left(\partial_r - c - \frac{1}{4}c_1 + i\epsilon_{ab}n^a\partial_r n^b \mp iR_0\Gamma^r\Gamma^\theta R_0^{-1}e^{\frac{1}{2}c_1r}\partial_r \ln \|\phi\|\right) + \Gamma^\theta\partial_\theta \right\} \Psi = 0, \quad (38)$$

where  $\hat{e}_\mu^\mu\Gamma^{\bar{\mu}}\hat{D}_\mu = \hat{e}_\mu^\mu\Gamma^{\bar{\mu}}(\partial_\mu - ieA_\mu)$  is the Dirac operator on the 4-dimensional braneworld in the background of the gauge field  $A_\mu$ . We are now ready to study the above Dirac equation



for 6-dimensional fluctuations, and write it in terms of 4-dimensional effective fields. Since  $\Psi$  is a 6-dimensional Weyl spinor we can represent it by [16]

$$\Psi = \begin{pmatrix} \Psi^{(4)} \\ 0 \end{pmatrix}, \quad (39)$$

where  $\Psi^{(4)}$  is a 4-dimensional Dirac spinor. Our choice for the 6-dimensional constant gamma matrices  $\Gamma^{\bar{M}}$ ,  $M = 0, 1, 2, 3, \bar{r}, \bar{\theta}$  are

$$\Gamma^{\bar{\mu}} = \begin{pmatrix} 0 & \gamma^{\bar{\mu}} \\ \gamma^{\bar{\mu}} & 0 \end{pmatrix}, \quad \Gamma^{\bar{r}} = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix}, \quad \Gamma^{\bar{\theta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (40)$$

where the  $\gamma^{\bar{\mu}}$  are the 4-dimensional constant gamma matrices and  $\gamma^5$  the 4-dimensional chirality matrix. Imposing the chirality condition  $\gamma^5 \Psi^{(4)} = +\Psi^{(4)}$ , the Dirac equation (38) can be written as

$$\left\{ e^{\frac{1}{2}cr} \hat{e}_{\bar{\mu}}^{\mu} \gamma^{\bar{\mu}} \hat{D}_{\mu} + \left( \partial_r - c - \frac{1}{4}c_1 + i\epsilon_{ab}n^a \partial_r n^b \pm R_0^{-2} e^{c_1 r} \partial_r \ln \|\phi\| \right) + iR_0^{-1} e^{\frac{1}{2}c_1 r} \partial_{\theta} \right\} \Psi^{(4)} = 0. \quad (41)$$

Now, from the equation of motion (41), we will search for the solutions of the form

$$\Psi^{(4)}(x, r, \theta) = \psi(x) \alpha(r) \sum e^{il\theta}, \quad (42)$$

where  $\psi(x)$  satisfies the massless 4-dimensional Dirac equation  $\hat{e}_{\bar{\mu}}^{\mu} \gamma^{\bar{\mu}} \hat{D}_{\mu} \psi = 0$ . For  $s$ -wave solution, Eq. (41) is reduced to

$$\left( \partial_r - c - \frac{1}{4}c_1 + i\epsilon_{ab}n^a \partial_r n^b \pm R_0^{-2} e^{c_1 r} \partial_r \ln \|\phi\| \right) \alpha(r) = 0. \quad (43)$$

The solution of this equation is given by

$$\alpha(r) \propto \exp \left\{ cr + \frac{1}{4}c_1 r - i \int^r dr \epsilon_{ab} n^a \partial_r n^b \mp R_0^{-2} \int^r dr e^{c_1 r} \partial_r \ln \|\phi\| \right\}. \quad (44)$$

So the fermionic zero mode reads

$$\Psi \propto \begin{pmatrix} \psi \\ 0 \end{pmatrix} \exp \left\{ cr + \frac{1}{4}c_1 r - i \int^r dr \epsilon_{ab} n^a \partial_r n^b \mp R_0^{-2} \int^r dr e^{c_1 r} \partial_r \ln \|\phi\| \right\}. \quad (45)$$

Now we wish to show that this zero mode is localized on the defect sitting around the origin  $r = 0$  under certain conditions. The condition for having localized 4-dimensional fermionic field is that  $\alpha(r)$  is normalizable. It is of importance to notice that normalizability

of the ground state wave function is equivalent to the condition that the “coupling” constant is nonvanishing.

Substituting the zero mode (45) into the Dirac action (31), the effective Lagrangian for  $\psi$  then becomes

$$\begin{aligned}\mathcal{L}_{eff}^{(0)} &= \int dr d\theta \sqrt{-g} \bar{\Psi} i \Gamma^M D_M \Psi \\ &= I_{1/2} \sqrt{-\hat{g}} \bar{\psi} i \hat{e}_\mu^\mu \gamma^\mu \hat{D}_\mu \psi,\end{aligned}\quad (46)$$

where

$$I_{1/2} \propto \int_0^\infty dr \exp \left( \frac{1}{2} cr \mp 2R_0^{-2} \int^r dr e^{c_1 r} \partial_r \ln \|\phi\| \right). \quad (47)$$

In order to localize spin 1/2 fermion in this framework, the integral (47) should be finite. From Eq. (47), one can see that whatever the form of  $A_r(r)$  is, the effective Lagrangian for  $\psi(x)$  has the same form. If the vortex background vanishes, this integral is obviously divergent for  $c > 0$  while it is finite for  $c < 0$ . If the vortex background does not vanish, the requirement that the integral (47) should be finite for  $c > 0$  is easily satisfied. For example, a simple choice is  $\|\phi\| = e^{\pm r}$  and  $c_1 > 0$ . These fermionic zero modes are generically normalizable on the brane in the self-dual vortex background if the integral  $I_{1/2}$  does not diverge.

Case II: the vacuum solution  $\|\phi\|^2 = v^2$ .

For the vacuum solution,  $\|\phi\|^2 = v^2$ , i.e.  $\phi = v e^{i\theta}$ , we have  $n^r = \cos \theta$ ,  $n^\theta = \sin \theta$ , and  $eA_r = 0$ ,  $eA_\theta = -1$ . The Dirac equation then becomes

$$\left\{ e^{\frac{1}{2}cr} \hat{e}_\mu^\mu \Gamma^{\bar{\mu}} \hat{D}_\mu + \Gamma^r \left( \partial_r - c - \frac{1}{4}c_1 \right) + \Gamma^\theta (\partial_\theta + i) \right\} \Psi = 0. \quad (48)$$

Repeating the deduction as the above case and imposing the chirality condition  $\gamma^5 \Psi^{(4)} = -\Psi^{(4)}$ , one can get the fermionic zero modes

$$\Psi \propto \begin{pmatrix} \psi \\ 0 \end{pmatrix} \exp \left\{ cr + \frac{1}{4}c_1 r - R_0^{-1} \int^r dr e^{\frac{1}{2}c_1 r} \right\}. \quad (49)$$

The effective Lagrangian for  $\psi(x)$  then becomes

$$\begin{aligned}\mathcal{L}_{eff}^{(0)} &= \int dr d\theta \sqrt{-g} \bar{\Psi}_0 i \Gamma^M D_M \Psi_0 \\ &= I_{1/2} \sqrt{-\hat{g}} \bar{\psi} i \gamma^\mu \hat{D}_\mu \psi,\end{aligned}\quad (50)$$

where

$$I_{1/2} \propto \int_0^\infty dr \exp \left( \frac{1}{2} cr - 2R_0^{-1} \int^r dr e^{\frac{1}{2} c_1 r} \right). \quad (51)$$

In order to localize spin 1/2 fermion on a string-like defect with the exponentially decreasing warp-factor (i.e.  $c > 0$ ) in this framework, the integral (51) should be finite. It is easy to see that the condition is  $c_1 > 0$ , i.e.

$$\begin{aligned} \Lambda &< -4\kappa_6^2 t_\theta, & \text{for } t_\theta > 0 \\ \Lambda &< \kappa_6^2 t_\theta. & \text{for } t_\theta < 0 \end{aligned} \quad (52)$$

This situation is a little different from the above case.

### B. Spin 3/2 fermionic field

Next we turn to spin 3/2 field, in other words, the gravitino. Let us start by considering the action of the Rarita-Schwinger gravitino field:

$$S_m = \int d^D x \sqrt{-g} \bar{\Psi}_M i \Gamma^{[M} \Gamma^N \Gamma^{R]} D_N \Psi_R, \quad (53)$$

where the square bracket denotes the anti-symmetrization, and the covariant derivative is defined with the affine connection  $\Gamma_{MN}^R = e_M^R (\partial_M e_N^{\bar{M}} + \omega_M^{\bar{M}\bar{N}} e_{N\bar{N}})$  by

$$D_M \Psi_N = \partial_M \Psi_N - \Gamma_{MN}^R \Psi_R + \omega_M^{\bar{M}\bar{N}} \Psi_{N\bar{N}}. \quad (54)$$

From the action (53), the equations of motion for the Rarita-Schwinger gravitino field are given by

$$\Gamma^{[M} \Gamma^N \Gamma^{R]} D_N \Psi_R = 0. \quad (55)$$

For simplicity, from now on we limit ourselves to the flat brane geometry  $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ . After taking the gauge condition  $\Psi_\theta = 0$  and  $A_\mu = 0$ , the non-vanishing components of the

covariant derivative are calculated as follows:

$$\begin{aligned}
D_\mu \Psi_\nu &= \partial_\mu \Psi_\nu - \frac{1}{2} c e^{-cr} \eta_{\mu\nu} \Psi_r + \frac{1}{4} c \Gamma_r \Gamma_\mu \Psi_\nu - i e A_\mu \Psi_\nu, \\
D_\mu \Psi_r &= \partial_\mu \Psi_r + \frac{1}{2} c \Psi_\mu + \frac{1}{4} c \Gamma_r \Gamma_\mu \Psi_r - i e A_\mu \Psi_r, \\
D_r \Psi_\mu &= \partial_r \Psi_\mu + \frac{1}{2} c \Psi_\mu - i e A_r \Psi_\mu, \\
D_r \Psi_r &= \partial_r \Psi_r - i e A_r \Psi_r, \\
D_\theta \Psi_\mu &= \partial_\theta \Psi_\mu + \frac{1}{4} c_1 \Gamma_r \Gamma_\theta \Psi_\mu - i e A_\theta \Psi_\mu, \\
D_\theta \Psi_r &= \partial_\theta \Psi_r + \frac{1}{4} c_1 \Gamma_r \Gamma_\theta \Psi_r - i e A_\theta \Psi_r, \\
D_\theta \Psi_\theta &= -\frac{1}{2} c_1 R_0 e^{-c_1 r} \Psi_r.
\end{aligned} \tag{56}$$

Again we represent  $\Psi_M$  as the following form

$$\Psi_M = \begin{pmatrix} \Psi_M^{(4)} \\ 0 \end{pmatrix}, \tag{57}$$

where  $\Psi_M^{(4)}$  is the 4D Rarita-Schwinger gravitino field.

Imposing the chirality condition  $\gamma^5 \Psi_\mu^{(4)} = +\Psi_\mu^{(4)}$ , and substituting Eqs. (56) and (57) into the equations of motion (55), we will look for the solutions of the form

$$\Psi_\mu^{(4)}(x, r, \theta) = \psi_\mu(x) u(r) \sum e^{i l \theta}, \tag{58}$$

$$\Psi_r^{(4)}(x, r, \theta) = \psi_r(x) u(r) \sum e^{i l \theta}, \tag{59}$$

where  $\psi_\mu(x)$  satisfies the following 4-dimensional equations  $\gamma^\mu \psi_\mu = \partial^\mu \psi_\mu = \gamma^{[\mu} \gamma^\nu \gamma^{\rho]} (\partial_\nu - i e A_\nu) \psi_\rho = 0$ . Then the equations of motion (55) reduce to

$$\left( \partial_r - \frac{1}{2} c - \frac{1}{4} c_1 - i e A_r(r) + e R_0^{-1} e^{\frac{1}{2} c_1 r} A_\theta(r) \right) u(r) = 0, \tag{60}$$

from which  $u(r)$  is easily solved to be

$$u(r) \propto \exp \left\{ \frac{1}{2} c r + \frac{1}{4} c_1 r + i e \int^r dr A_r(r) - e R_0^{-1} \int^r dr e^{\frac{1}{2} c_1 r} A_\theta(r) \right\}. \tag{61}$$

In the above we have considered the  $s$ -wave solution and  $\psi_r = 0$ .

Let us substitute the zero mode (61) into the Rarita-Schwinger action (53). It turns out that the effective Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_{eff} &= \int dr d\theta \sqrt{-g} \bar{\Psi}_M i \Gamma^{[M} \Gamma^N \Gamma^{R]} D_N \Psi_R \\
&= I_{3/2} \bar{\psi}_\mu i \gamma^{[\mu} \gamma^\nu \gamma^{\rho]} (\partial_\nu - i e A_\nu) \psi_\rho,
\end{aligned} \tag{62}$$

where the integral  $I_{3/2}$  is defined as

$$I_{3/2} \propto \int_0^\infty dr \exp \left( \frac{1}{2}cr - 2eR_0^{-1} \int^r dr e^{\frac{1}{2}c_1 r} A_\theta(r) \right). \quad (63)$$

As in the above subsection, to illustrate how the vortex background affects the fermionic zero modes, we first discuss the simple case that the Higgs field  $\phi$  is only relative to  $r$ , and then solve the general Dirac equation for the vacuum Higgs field solution  $\|\phi\| = v$ .

Case I:  $\phi = \phi(r) = \phi^1(r) + i\phi^2(r)$ .

In this case, Eq.(10) reduces to Eqs. (36) and (37), and the integral  $I_{3/2}$  can be expressed as

$$I_{3/2} \propto \int_0^\infty dr \exp \left( \frac{1}{2}cr \mp 2R_0^{-2} \int^r dr e^{c_1 r} \partial_r \ln \|\phi\| \right). \quad (64)$$

It is easy to see that this expression is equivalent to  $I_{1/2}$  in (47) up to an overall constant factor so we encounter the same result as in the corresponding case for spin 1/2 field. So the zero modes for spin 3/2 field are generically normalizable on the brane in the self-dual vortex background if the integral  $I_{3/2}$  (64) does not diverge.

Case II: the vacuum solution  $\|\phi\|^2 = v^2$ .

In this case,  $eA_r = 0$ ,  $eA_\theta = -1$ . Again, changing the chirality condition to  $\gamma^5 \Psi_\mu^{(4)} = -\Psi_\mu^{(4)}$ , the integral  $I_{3/2}$  takes the form

$$I_{3/2} \propto \int_0^\infty dr \exp \left( \frac{1}{2}cr - 2R_0^{-1} \int^r dr e^{\frac{1}{2}c_1 r} \right), \quad (65)$$

which is equivalent to  $I_{1/2}$  in (51) up to an overall constant factor so it is also finite for  $c > 0$  and  $c_1 > 0$ .

## V. DISCUSSIONS

Using the generalized Abelian Higgs model and  $\phi$ -mapping theory, we investigate the self-dual vortex on an extra two-dimensional curved Riemann surface, and obtain the inner topological structure of the self-dual vortex. Under the gravity and vortex backgrounds, we have investigated the possibility of localizing the spin 1/2 and 3/2 fermionic fields on a brane with the exponentially decreasing warp factor. We first give a brief review of a string-like defect solution to Einstein's equations with sources, then check localization of fermionic fields on such a string-like defect with the background of self-dual vortex from the viewpoint

of field theory. It has been found that the vortex background affects the fermionic zero modes, and that spin 1/2 and 3/2 fields can be localized on a defect with the exponentially decreasing warp factor if self-dual vortex and gravitational backgrounds are considered.

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